Uses of Taylor Series

1. Paraxial Approximation in Optics
2. Numerical Differentiation
   a. Expansions in time
   b. Expansions in space
3. Numerical integration methods
4. Derivation of continuinity equation
   a. conservation of mass
   b. Conservation of charge
   c. conservation of probabity.

Taylor series is polynomial representation of function.

$$f(x) = \sum_{i=0}^{\infty} a_i (x-x_e)^n_i = a_0 + a_1(x_1-x_e) + a_2(x_2-x_e)^2 + \ldots$$

How to determine coefficents a_i ?

$$a_0 = f(x)\Big|_{x=x_e}$$

take differential

$$\frac{df(x)}{dx} = \sum_{i=1}^{\infty} a_i n (x-x_e)^{n-1}$$

$$a_1 = \frac{df(x)}{dx\, n}\Big|_{x=x_e}$$

Can repeat process finding

$$a_i = \frac{d^n f(x)}{n!\, dx^n}\Big|_{x_e=x_e}$$

Mcclaurin Series

There are other basis functions we could expand into such as sinusoids.

Advantage of polynomial expansion?

Polynomials are easily integrated and diffentiated.

Fourier Series is another expansion using sinusoids

$$f(x) = a_0 + \sum_{i=1}^{\infty} a_n \sin(nx) + \sum_{i=1}^{\infty} b_n \cos(nx)$$

A sinusoid basis is useful because sinusoids are eigenfunctions of linear time invarient systems. Example LTI system is network of resistors, capacitors, and inductors.

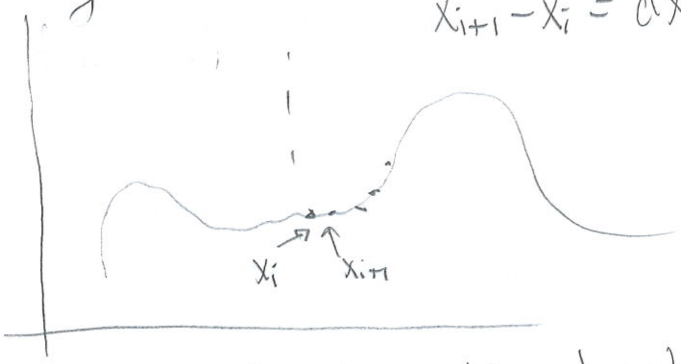
Back to taylor series.

Much of the usefulness in numerical methods is taylor series let's us approximate the differential.

1st. Note taylor series lets us approximate on near by region using differentials

$x_{i+1} - x_i = dx = h$

consider

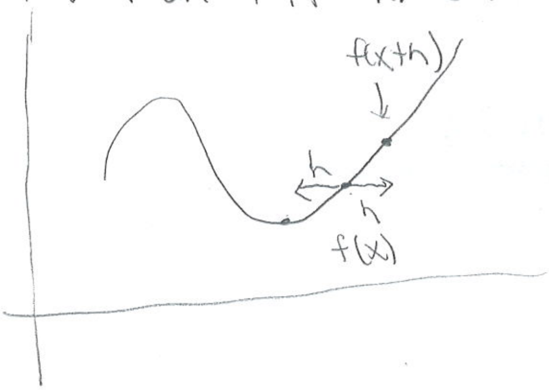


we can approximate $f(x_{i+1})$ with taylor expansion about x_i

$$f(x_{i+1}) = f(x_i) + f'(x_i)\cdot(x_{i+1} - x_i) + \frac{f''(x_i)}{2}(x_{i+1} - x_i)^2$$

we can rearange

Numerical Approx. of derivatives



Simple approximation is

$$\frac{d}{dx} f(x) \simeq \frac{f(x+h) - f(x)}{h} \quad ; \text{ forward difference}$$

what is error. Estimate w/ taylor series ?

$$f(x+h) = f(x) + h \frac{\partial}{\partial x} f(x) + h^2 \frac{\partial}{\partial x} f(x) + \mathcal{O}(h^3)$$

$$\frac{\partial}{\partial x} f(x) = \frac{f(x+h) - f(x) - \frac{h^2}{2} \frac{\partial^2}{\partial x^2} f(x)}{h}$$

$$\frac{\partial}{\partial x} f(x) = \frac{f(x+h) - f(x)}{h} + h \frac{\partial^2}{\partial x^2} (f(x))$$

$$f(x-h) = f(x) - h \frac{\partial}{\partial x} f(x) + \frac{h^2}{2} \frac{\partial^2}{\partial x^2} f(x)$$

$$\frac{\partial}{\partial x} f(x) = \frac{f(x) - f(x-h) + \frac{h^2}{2} \frac{\partial^2}{\partial x^2} f(x)}{h}$$

what is error for average

$$\frac{\frac{\partial}{\partial x} f(x) + \frac{\partial}{\partial x} f(x)}{2} = \frac{f(x+h) - f(x-h)}{2h} + \mathcal{O}(h^2)$$

Taylor series expansions in 3 dimensions

$$f(x,y,z) = f(x_0,y_0,z_0) + (x-x_0)\frac{\partial}{\partial x} f(x_0,y_0,z_0) + (y-y_0)\frac{\partial}{\partial y} f(x_0,y_0,z_0) + (z-z_0)\frac{\partial}{\partial z} f(x_0,y_0,z_0)$$

S.O.T. $\longrightarrow$

$$+\frac{1}{2}\Big((x-x_0)^2 \frac{\partial^2}{\partial x^2} f(x_0,y_0,z_0) + (x-x_0)(y-y_0)\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x_0,y_0,z_0) +$$

Notice x-ing terms

$$(x-x_0)(z-z_0)\frac{\partial}{\partial x}\frac{\partial}{\partial z} f(x_0,y_0,z_0) + (x-x_0)(y-y_0)\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x_0,y_0,z_0)$$

$$(y-y_0)^2\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x_0,y_0,z_0) + (y-y_0)(z-z_0) f_{yz}(x_0,y_0,z_0) +$$

$$(z-z_0)(x-x_0) f_{zx}(x_0,y_0,z_0) + (z-z_0)(y-y_0) f_{zy}(x_0,y_0,z_0)$$

$$+ (z-z_0)^2 f_{zz}(x_0,y_0,z_0) \Big) + R_3(x,y,z)$$

Commutation of operators

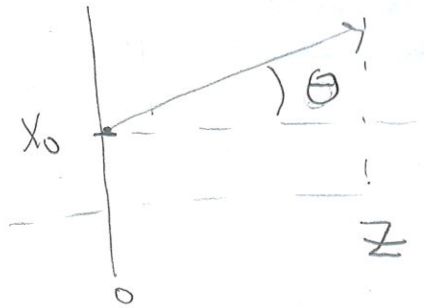
$$\text{is } \frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial x} = 0 ?$$

$$[P_x, P_y] \overset{?}{=} 0 \qquad p = i\hbar\frac{\partial}{\partial x}$$

$$P_x P_y - P_y P_x \overset{?}{=} 0$$

Pariaxil Approximation.

Rayoptics Model Light as



$$\begin{bmatrix} X \\ \theta \end{bmatrix}$$

new position of ray is $X_1 = X_0 + Z \cdot \tan(\theta)$

would like to as matrix

$$\begin{bmatrix} X_Z \\ \theta_Z \end{bmatrix} = \begin{bmatrix} 1 & ? \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ \theta_1 \end{bmatrix}$$

$? = \tan(\theta)$

not nice try expanding $\tan\theta$ around zero, small angle approx.

$$\tan(\theta) = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \quad \theta \text{ in radians}$$

$$\tan(\theta) = \theta$$

$$\begin{bmatrix} X_Z \\ \theta_Z \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ \theta_1 \end{bmatrix}$$

Can comeup with similar formulas for lens.

Additionaly, often $K = \frac{2\pi}{\lambda}$

$$\vec{K} = K_x \hat{x} + K_y \hat{y} + K_z \hat{z}$$

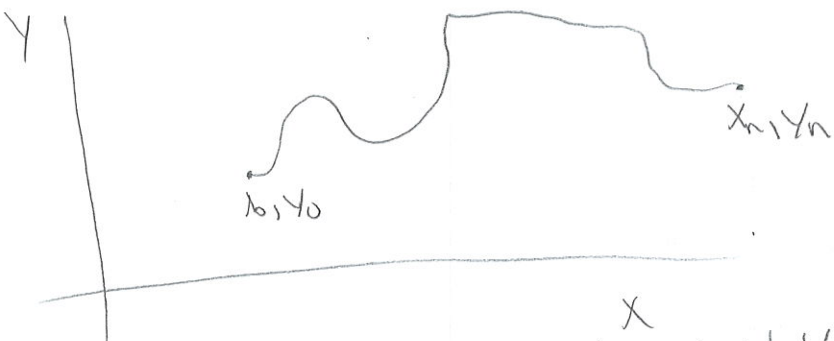
$$|K|^2 = K_x^2 + K_y^2 + K_z^2$$

Solve $K_z = \sqrt{|K|^2 - (K_x^2 - K_y^2)}$ enters $e^{-j K_z Z}$ in s.f. domain

2. Numerical integration of ODEs
Solutions of
o.d.e. $\frac{dy}{dx} = f(x,y)$

What are we solving for?

Given initial conditions x_0, y_0 what is y_n @ x_n.



Let's taylor series expand y_n at y_n

$$y_{n+1} = y_n + h \cdot \frac{dy}{dx}\bigg|_{y_n} + \frac{1}{2}\frac{d^2y}{dx^2}\bigg|_{y_n} \cdot h^2$$

$$y_{n+1} = y_n + f(x_1, y_1)\,h + O(2)$$

$y_{n+1} = y_n + f(x_1, y_1)\,h$ is euler's method.

Expansions of space. Derivations of continuity equation.

Thermodynamic Laws. { 0. Conservation of mass
1. Conservation of Energy }
2. Entropy increases
3. T=0 on kelvin scale.

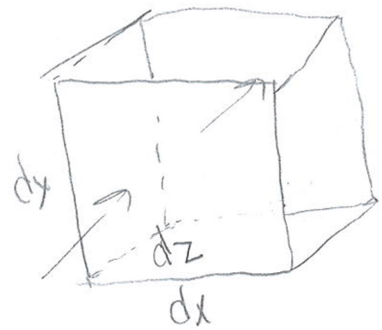
continuity equation.

Continuity of mass: $-\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \vec{V})$ ← velocity

Continuity of charge (consistant w/ Maxwell) $-\frac{\partial q}{\partial t} = \nabla \cdot \vec{J}$ ← current flux I/A

Continuity of mass is derived through taylor expansion

consider a cube containg a "liquid" with density ρ



mass inside volume is

$\rho\, dx\, dy\, dz$, $dx\, dy\, dz = d\mathcal{V}$

but what is the chang in mass over time

$\frac{\partial \rho\, d\mathcal{V}}{\partial t}$? must be equal to the mass in minus the mass out

$= \sum_{in} \rho A_{in} V_{in} - \sum \rho A V_{out}$

for x face $= \rho V_{x_i}\, dy\, dz - \rho V_{x+dx}\, dy\, dz$

y face $= \rho V_y\, dx\, dz - \rho V_{y+dy}\, dx\, dz$

$= \rho V_z\, dx\, dy - \rho V_{z+dz}\, dy\, dz$

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Can write V_{x+dx} as taylor series expansion

$$V_{x+dx} = V_x + \frac{\partial V_x}{\partial x} \cdot \partial x$$

similarly

$$V_{y+dy} = V_y + \frac{\partial V_y}{\partial y} \cdot \partial y$$

putting it together for the 3 faces

$$\frac{\partial \rho \, \partial V}{\partial t} = \rho \frac{\partial V_x}{\partial x} \cdot dx + \rho \frac{\partial V_y}{\partial y} \cdot dy + \rho \frac{\partial V_z}{\partial z} \cdot dz$$

define $\nabla \triangleq \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$

and $\vec{V} = \hat{i} V_x + \hat{j} V_y + \hat{k} V_z$

$$\frac{\partial \rho \, \partial V}{\partial t} = -\rho (\nabla \cdot \vec{V}) \, dx \, dy \, dz$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{V})$$